Quaternions and their Applications

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INTRODUCTION

Sir William Rowan Hamilton

(1805-1865)

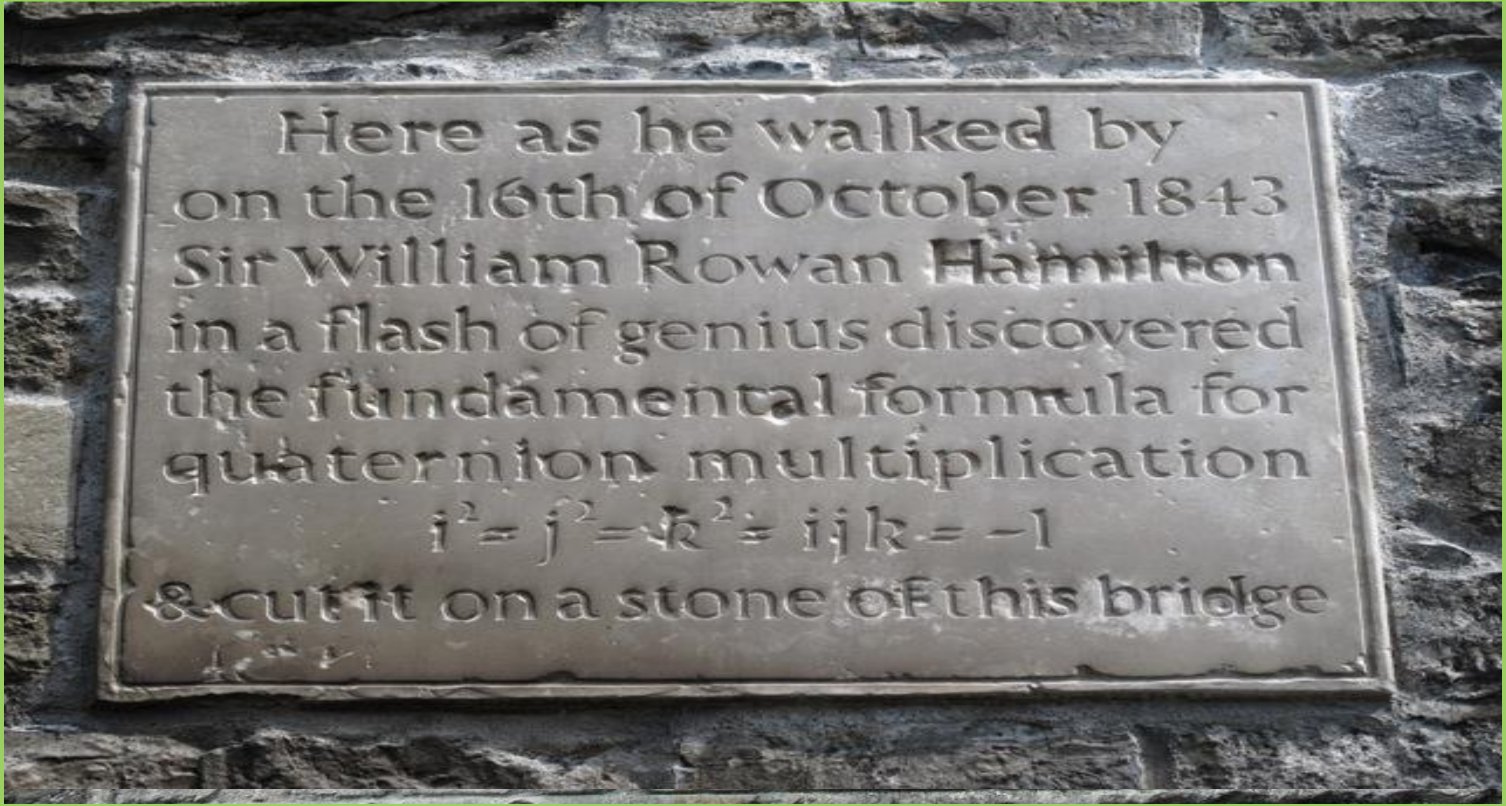
Quaternions were introduced by Irish mathematician Sir William Rowan Hamilton 1843.

Multiplication of quaternions was introduced by Hamilton in 1843 . According to the story, he had struggled with the problem of defining multiplication of vectors in $\mathbf{R}^3$ since 1833, and his family took a great interest in this. Each morning at breakfast, his boys would ask “Well, Papa, can you multiply triplets?” (meaning vectors in **R**3) and would receive the sad reply

“No I can only add and subtract them.

Then when strolling with his wife by Brougham Bridge in Dublin one day, it suddenly occurred to him that all the difficulties would disappear if he used quadruples-that is vectors in $\mathbf{R}^4$.

Brougham Bridge



Hamilton then promptly carved this equation into the side of the nearby Brougham Bridge called.

Hamilton also described a quaternion as an ordered quadruple (4-tuple) of real numbers, and described the first coordinate as the 'scalar' part, and the remaining three as the 'vector' part. If two quaternions with zero scalar parts are multiplied, the scalar part of the product is the negative of the dot product of the vector parts, while the vector part of the product is the cross product. But the significance of these was still to be discovered. Hamilton proceeded to popularize quaternions with several books, the last of which, *Elements of Quaternions*, had 800 pages and was published shortly after his death.

While the complex numbers are obtained by adding the element *i* to the real numbers which satisfies $i^2 = -1$, the quaternions are obtained by adding the elements *i*, *j* and *k* to the real numbers which satisfy the following relations.

$$i^2 = j^2 = k^2 = ijk = -1$$

If the multiplication is assumed to be associative (as indeed it is), the following relations follow directly:

$$\begin{aligned} ij &= k & ji &= -k \\ jk &= i & kj &= -i \\ ki &= j & ik &= -j \end{aligned}$$

(these are derived in detail below).

Every quaternion is a real linear combination of the **basis quaternions** *1*, *i*, *j*, and *k*, i.e. every quaternion is uniquely expressible in the form $a + bi + cj + dk$ where *a*, *b*, *c*, and *d* are real numbers. In other words, as a vector space over the real numbers, the set *H* of all quaternions has dimension *4*, whereas the complex number plane has dimension *2*.
The *conjugate* of the quaternion is defined as

$$\bar{q} = a - bi - cj - dk$$

and the norm of *q* is the non-negative real number defined by

$$Nq = q\bar{q} = a^2 + b^2 + c^2 + d^2$$

Quaternions second definition:

$$p = S_p + V_p = a + bi + cj + dk$$

where $S_p = a$ scalar, $V_p = bi + cj + dk$ vector part of *p*.

Addition and products

Quaternion addition
$p + q$:

$$p+q=(S_p+S_q)+(V_p+V_q)$$

Like complex numbers, vectors, and matrices, the addition of two quaternions is equivalent to summing the elements together:

Addition follows all of the commutativity and associativity rules of real and complex number.

Quaternion multiplication *pq* :

The usual non-commutative multiplication between two quaternions is termed the Product. This product has been described briefly above. The complete form is described below:

$$pq = (S_p + V_p)(S_q + V_q) = S_p S_q - V_p . V_q + S_p V_q + S_q V_p + V_p \times V_q$$

Due to the non-commutative nature of the quaternion multiplication, *pq* is not equivalent to *qp*.

Quaternion dot-product $p \cdot q$:

The dot-product is also referred to as the Euclidean inner product, and is equivalent to a 4-vector dot product. The inner product is the sum of the quantity of each element of *p* multiplied by each element of *q*. It is a commutative product between quaternions, and returns a scalar quantity.

$$p.q = at + bx + cy + dz, \quad q = t + xi + yj + zk$$

The dot-product can be rewritten using the quaternion product:

$$p.q = \frac{\overline{p}q + \overline{q}p}{2}$$

Quaternion reciprocal p^{-1}

The inverse of a quaternion is defined in a way that $p^{-1}p = 1$. It is defined above in the definition section, under properties (note the difference in variable notation). It is formed the same way that the complex inverse is found:

$$p^{-1} = \frac{\overline{p}}{Np}$$

The dot product of a quaternion is a scalar. The division of a quaternion by a scalar is equivalent to multiplication by the scalar inverse, such that each element of the quaternion is divided by the divisor.

Quaternion division $p^{-1}q$

The non-commutativity of quaternions allows for two divisions of numbers $p^{-1}q$ and qp^{-1}. This means that the notation of q/p cannot be used unless p is a scalar only.

Quaternion modulus $|p|$:

The absolute value of a quaternion is the scalar quantity that determines the length of the quaternion from the origin.

$$\|p\| = \sqrt{p.p} = \sqrt{p\,\overline{p}} = \sqrt{a^2 + b^2 + c^2 + d^2}$$

By using the distance function, $d(p,q) = \|p - q\|$ the quaternions form a metric space (isometric to the usual Euclidean metric on $\boldsymbol{R}^4$) and the arithmetic operations are continuous. We also have $\|pq\| = \|p\|\|q\|$ for all quaternions *p* and *q*.

There are only two unit sphere which have a group structure. This spheres are S^1 and S^3.

Theorem. The set of unit quaternion $S^3 = \{q \in H \;/\; q\bar{q} = 1\}$ is a Lie group.

REPRESENTİNG QUATERNİONS BY MATRİCES

There are at least two ways of representing quaternions as matrices, in such a way that quaternion addition and multiplication correspond to matrix addition and matrix multiplication (i.e., quaternion-matrix homomorphisms). One is to use 2×2 complex matrices, and the other is to use 4×4 real matrices.

In the first way, the quaternion *p=a + bi + cj + dk* is represented as

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$$

This representation has several nice properties.

- Complex numbers ($c = d = 0$) correspond to diagonal matrices.
- The square of the absolute value of a quaternion is the determinant of the corresponding matrix.
- The conjugate of a quaternion corresponds to the conjugate transpose of the matrix.
- Restricted to unit quaternions, this representation provides the isomorphism between S^3 and $SU(2)$. The latter group is important in quantum mechanics when dealing with spin; see also Pauli matrices.

In the second way, the quaternion $p=a + bi + cj + dk$ is represented as

$$A_p = \begin{bmatrix} a & -b & d & -c \\ b & a & -c & -d \\ -d & c & a & -b \\ c & d & b & a \end{bmatrix}$$

In this representation, the conjugate of a quaternion corresponds to the transpose of the matrix. The fourth power of the absolute value of a quaternion is the determinant of the corresponding matrix.

QUATERNİON ROTATİON

It is well known that the vector product is related to rotation in space. The goal then is to find a formula which expresses rotation in Euclidean space E^3 using quaternion multiplication, similar to the formula for a rotation in E^2 using complex multiplication,

$$f : \mathbb{C} \to C$$

$$z \to f(z) = zw \qquad\qquad z, w \in C$$

where

$$z = e^{\alpha i}$$

is used for rotation by an angle α.

ROTATİONS İN R^3

Let $p = S_p + V_p$ be a unit quaternion, and consider the function

$$\Phi : R^3 \to R^3$$
$$x \to \Phi(x) = pxp^{-1}$$

where p^{-1} is the multiplicative inverse of *p* and *x* is a vector, considered as a quaternion with zero real part.

$$real\left(pxp^{-1}\right) = real\left(pp^{-1}x\right) = real\left(x1\right) = 0$$

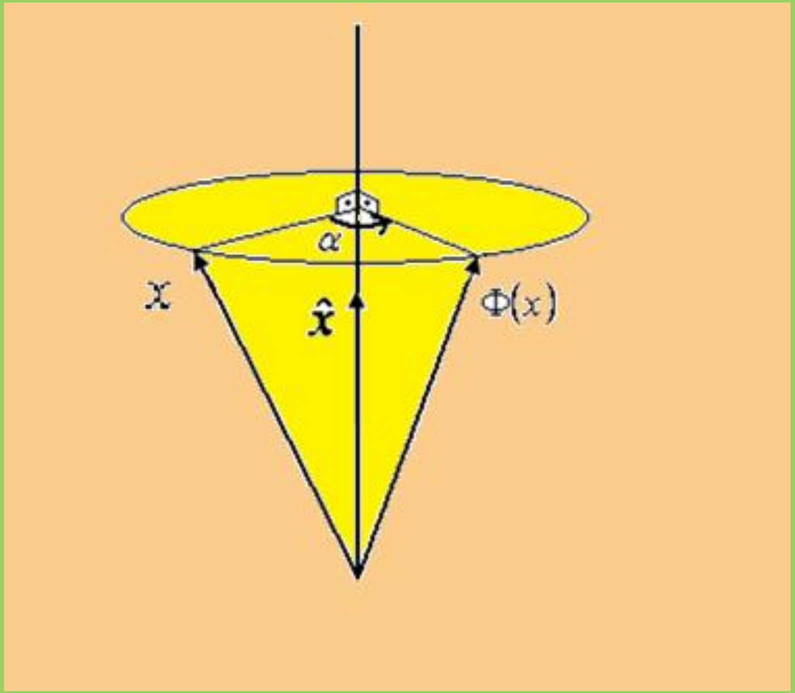
Furthermore, Φ is a R-linear and we have $\Phi(x) = x$ if and only if x and the imaginary part V_p of p are collinear (because $\Phi(x) = x$ means $x\,p = p\,\mathbf{x}$). Hence Φ is a rotation whose axis of rotation passes through the origin and is given by the vector V_p.

Note that even then p and $-p$ represent the same rotation.

To summarize, a counterclockwise rotation through an angle α about an axis $\hat{x}$ can be represented

$$p = \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\,\hat{x}$$

where $\hat{x}$ is a unit *vector.*



A counterclockwise rotation through an angle α about an axis $\hat{x}$ in R^3

AN EXAMPLE

Let us consider the rotation Φ around the axis **u** = *i* + *j* + *k*, with an rotation angle of 120°—i.e., $^{2\pi}/_3$ radians.

$$\alpha = \frac{2\pi}{3} = 120°$$

The length of ***u*** is , the half angle is $^{\pi}/_3$ (60°) with cosine $^1/_2$ (cos 60° = 0.5) and sine $^{\sqrt{3}}/_2$ (sin 60° = 0.866). We are therefore dealing with a conjugation by the unit quaternion

$$q = \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\hat{x} = \frac{1}{2} + \frac{\sqrt{3}}{2}\hat{x}, \qquad \hat{x} = \frac{u}{\|u\|}$$

QUATERNİONS VERSUS OTHER REPRESENTATİONS OF ROTATİONS

The representation of a rotation as a quaternion (4 numbers) is more compact than the representation as an orthogonal matrix (9 numbers). Furthermore, for a given axis and angle, one can easily construct the corresponding quaternion, and conversely, for a given quaternion one can easily read off the axis and the angle. Both of these are much harder with matrices or Euler angles.

In computer games and other applications, one is often interested in "smooth rotations," meaning that the scene should slowly rotate and not in a single step. This can be accomplished by choosing a curve such as the spherical linear interpolation in the quaternions, with one endpoint being the identity transformation *1* (or some other initial rotation) and the other being the intended final rotation. This is more problematic with other representations of rotations.

When composing several rotations on a computer, rounding errors necessarily accumulate. A quaternion that's slightly off still represents a rotation after being normalized—a matrix that's slightly off need not be orthogonal anymore and therefore is harder to convert back to a proper orthogonal matrix.

Since Φ is a linear function, the orthogonal matrix corresponding to a rotation by the unit quaternion *p = a + bi + cj + dk* (with $\|p\| = 1$) is given by

$$M = \begin{bmatrix} a^2 + b^2 - c^2 - d^2 & -2ad + 2bc & 2ac + 2bd \\ 2ad + 2bc & a^2 - b^2 + c^2 - d^2 & 2cd - 2ab \\ 2bd - 2ac & 2ab + 2cd & a^2 - b^2 - c^2 + d^2 \end{bmatrix}$$

of rotations.

In ordinary three dimensional space, a coordinate rotation can be described by means of Euler angles. It can also be described by means of quaternions (see below), an approach which is similar to the use of vector calculus.

Another way is to multiply by a matrix M, which will rotate space by an angle θ around a unit vector $\mathbf{v}=(x, y, z)$, or, alternatively, provides the formulas for converting coordinates if the coordinate axes rotate in opposite direction:

$$M(v,\theta) = \begin{bmatrix} cos\theta + (1-cos\theta)x^2 & (1-cos\theta)xy - (sin\theta)z & (1-cos\theta)xz + (sin\theta)y \\ (1-cos\theta)yx + (sin\theta)z & cos\theta + (1-cos\theta)y^2 & (1-cos\theta)yz - sin\theta x \\ (1-cos\theta)zx - (sin\theta)y & (1-cos\theta)zy + (sin\theta)x & cos\theta + (1-cos\theta)z^2 \end{bmatrix}$$

or

$$M(v,\theta) = I_3 + (sin\theta)S + (1-cos\theta)S^2$$

where

$$S = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

is a skew symmetric matrix.

REFLECTİON İN R^3

If we consider the linear map:

$$\Psi : R^3 \to R^3$$
$$w \to \Psi(w) = \vec{q} w \vec{q},$$

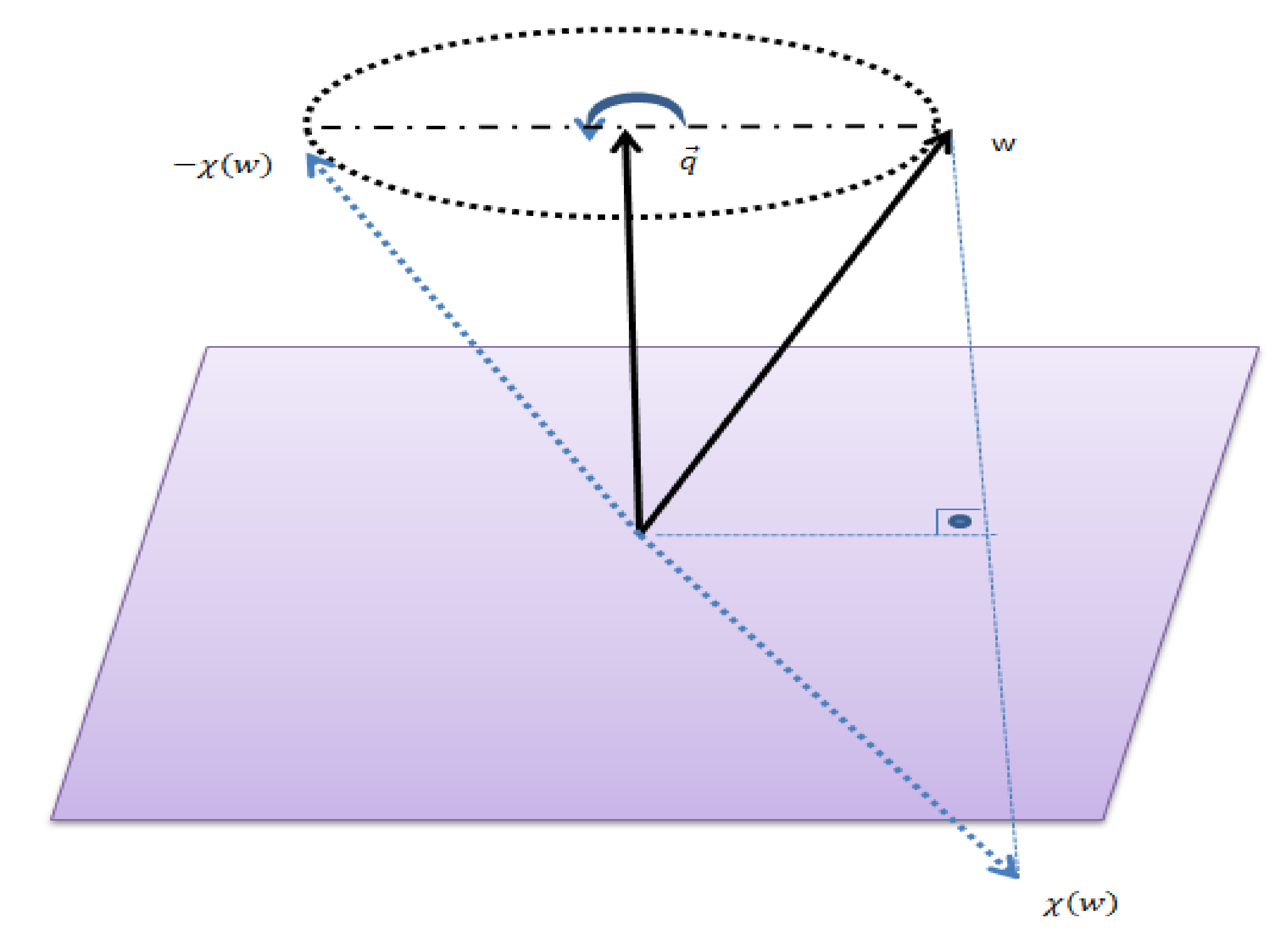
where $\vec{q} = bi + cj + dk$ is a unit vector quaternion.

Then the matrix representation of this map N :

$$N = \begin{bmatrix} -b^2+c^2+d^2 & -2bc & -2bd \\ -2bc & b^2-c^2+d^2 & -2cd \\ -2bd & -2cd & b^2+c^2-d^2 \end{bmatrix}$$

where $\det(N) = -1$ and the map Ψ is not orientation preserving and represent a reflection (as we saw above it is a reflections of w is the plane with normal $\vec{q}$).

Then, $\Psi(w) = N.w.$ The matrix corresponding to this linear transformation ($\Psi : R^3 \to R^3$) is reflection matrix.



$-\chi(w)$
$\vec{q}$
w
$\chi(w)$

Then, $-\chi(w) = -qwq$. The reflection according to the line which $\vec{q}$ is the normal is the same with 180° rotation around $\vec{q}$.

$$
\begin{aligned}
-\chi(w) &= -qwq \\
&= (I_3 + 2S^2)w = -Nw
\end{aligned}
$$

DUAL NUMBERS AND DUAL QUATERNIONS

A dual number A has the form $a+\varepsilon a^{*}$ where a and a^{*} are real numbers and ε is the dual symbol subjected to the rules

$$\varepsilon \neq 0, \qquad 0\varepsilon = \varepsilon 0 = 0, \qquad 1\varepsilon = \varepsilon 1 = \varepsilon, \qquad \varepsilon^{2} = 0.$$

A dual quaternion Q is written as

$$Q = A_0 + A_1 i + A_2 j + A_3 k.$$

The three dual quaternionic units (*i*, *j*, and *k*) are orthogonal unit vector with respect to scalar product defined below. *i*, *j*, and *k* are identified as an orthogonal triad of unit vectors in Euclidean 3-space. It is useful, therefore, to define the following terms:

Dual number part of Q:

$$S_Q = A_0$$

Dual vector part of Q:

$$V_Q = A_1 i + A_2 j + A_3 k$$

Hamiltonian conjugate of Q:

$$\bar{Q} = A_0 - (A_1 i + A_2 \mathrm{j} + A_1 k) = S_Q - V_Q = \bar{q} + \varepsilon \bar{q}^*.$$

The dual quaternion multiplication is, in general, not commutative. If Q and P be the two dual quaternions and let $R=QP$, then R is given by

$$QP = (S_Q + V_Q)(S_P + V_P) = S_Q S_P - V_Q . V_P + S_Q V_P + S_P V_Q + V_Q \times V_P$$

where $P = B_0 + B_1 i + B_2 j + B_3 k = p + \varepsilon p^*$, p and p^* are real quaternions.

Norm of Q:

$$N_Q = Q\overline{Q} = \overline{Q}Q$$
$$= A_0^2 + A_1^2 + A_2^2 + A_3^2$$

Reciprocal of Q:

$$Q^{-1} = \frac{\overline{Q}}{Q\overline{Q}}$$

where $(N_q)^2 \neq 0.$

Unit quaternion:

$$N_Q = 1.$$

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HAMILTON OPERATORS AND THEIR PROPERTIES

In this section, two new operators $\overset{+}{H}$ and $\overset{-}{H}$, called Hamilton's operators, are defined and its properties are discussed. If q is a quaternion, then Hamilton operators $\overset{+}{H}$ and $\overset{-}{H}$ are, respectively, defined as

$$\overset{+}{H}(q)=\begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}$$

and

$$\bar{H}(q)=\begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix}$$

A direct consequence of the above operators is the following identities:

$$\overset{+}{H}(1) = \overset{-}{H}(1) = I,$$

$$\overset{+}{H}(i) = E_1; \quad \overset{+}{H}(j) = E_2; \quad \overset{+}{H}(k) = E_3,$$

$$\overset{-}{H}(i) = F_1; \quad \overset{-}{H}(j) = F_2; \quad \overset{-}{H}(k) = F_3.$$

where I is a 4x4 identity matrix. Note that the properties of E_n and F_n (n=1,2,3) are identical to that of quaternionic unit i, j, k.

Note that the properties of E_n and F_n (n=1,2,3) are identical to that of quaternionic unit i, j, k.

In R^4, the basis of quaternions can be given by

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},\ E_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},\ E_4 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Since $\overset{+}{H}$ and $\overset{-}{H}$ are linear in its elements, it follows that

$$\overset{+}{H}(q) = a_0\,\overset{+}{H}(1) + a_1\,\overset{+}{H}(i) + a_2\,\overset{+}{H}(j) + a_3\,\overset{+}{H}(k)$$
$$= a_o I + a_1 E_1 + a_2 E_2 + a_3 E_3,$$

$$\overset{-}{H}(q) = a_0\,\overset{-}{H}(1) + a_1\,\overset{-}{H}(i) + a_2\,\overset{-}{H}(j) + a_3\,\overset{-}{H}(k)$$
$$= a_o I + a_1 F_1 + a_2 F_2 + a_3 F_3,$$

We can give $\overset{+}{H}$ and $\overset{-}{H}$ operators for dual quaternions. Thus,

$$\overset{+}{H}(Q) = \overset{+}{H}(q) + \epsilon\,\overset{+}{H}(q^*)'$$

$$\overset{-}{H}(Q) = \overset{-}{H}(q) + \epsilon\,\overset{-}{H}(q^*)$$

.

Using the definitions of $\overset{+}{H}$ and $\overset{-}{H}$, the multiplication of the two dual quaternions Q and *P* is given by

$$R = \overset{+}{H}(Q)P = \overset{-}{H}(P)Q$$

Theorem. If Q and P are dual quaternions and λ is a real number and the operators $\overset{+}{H}$ and $\overset{-}{H}$ hold:

- $Q = P \Leftrightarrow \overset{+}{H}(Q) = \overset{+}{H}(P) \Leftrightarrow \overset{-}{H}(Q) = \overset{-}{H}(P).$
- $\overset{+}{H}(Q+P) = \overset{+}{H}(Q) + \overset{+}{H}(P),\ \overset{-}{H}(Q+P) = \overset{-}{H}(Q) + \overset{-}{H}(P).$

- $\overset{+}{H}(\lambda Q) = \lambda \overset{+}{H}(Q)$, $\overset{-}{H}(\lambda Q) = \lambda \overset{-}{H}(Q)$.
- $\overset{+}{H}(QP) = \overset{+}{H}(Q)\overset{+}{H}(P)$, $\overset{-}{H}(QP) = \overset{-}{H}(P)\overset{-}{H}(Q)$.
- $\overset{+}{H}(\overline{Q}) = \overset{+}{H}(Q)^T$, $\overset{-}{H}(\overline{Q}) = \overset{-}{H}(Q)^T$.
- $\overset{+}{H}(Q^{-1}) = \overset{+}{H}(Q)^{-1}$, $\overset{-}{H}(Q^{-1}) = \overset{-}{H}(Q)^{-1}$, $(N_Q)^2 \neq 0$.
- $\mathrm{tr}\left[\overset{+}{H}(Q)\right] = 4A_0$, $\mathrm{tr}\left[\overset{-}{H}(Q)\right] = 4A_0$.
- $\det\left[\overset{+}{H}(Q)\right] = (N_Q)^2$, $\det\left[\overset{-}{H}(Q)\right] = (N_Q)^2$.

Theorem. Matrices generated by operators $\overset{+}{H}$ and $\overset{-}{H}$ commute, or mathematically this can be stated as

$$\overset{+}{H}(Q)\,\overset{-}{H}(P)=\overset{-}{H}(P)\,\overset{+}{H}(Q).$$

SCREW OPERATORS

Let $\vec{A}$ and $\vec{B}$ be unit dual vectors in ID^3 the quaternion product of these two dual vectors is given by

$$\vec{A}\vec{B} = -\left(\vec{A}.\vec{B}\right) + \vec{A} \times \vec{B} \qquad (1)$$

We can rewrite the expression (1) as:

$$\vec{A}\vec{B} = -\cos\Phi + \frac{\vec{A} \times \vec{B}}{\left\|\vec{A} \times \vec{B}\right\|}\left\|\vec{A} \times \vec{B}\right\| \qquad (2)$$

$$= -\cos\Phi + \vec{S}\sin\Phi \qquad (3)$$

where $\vec{S} = \dfrac{\vec{A} \times \vec{B}}{\left\| \vec{A} \times \vec{B} \right\|}$

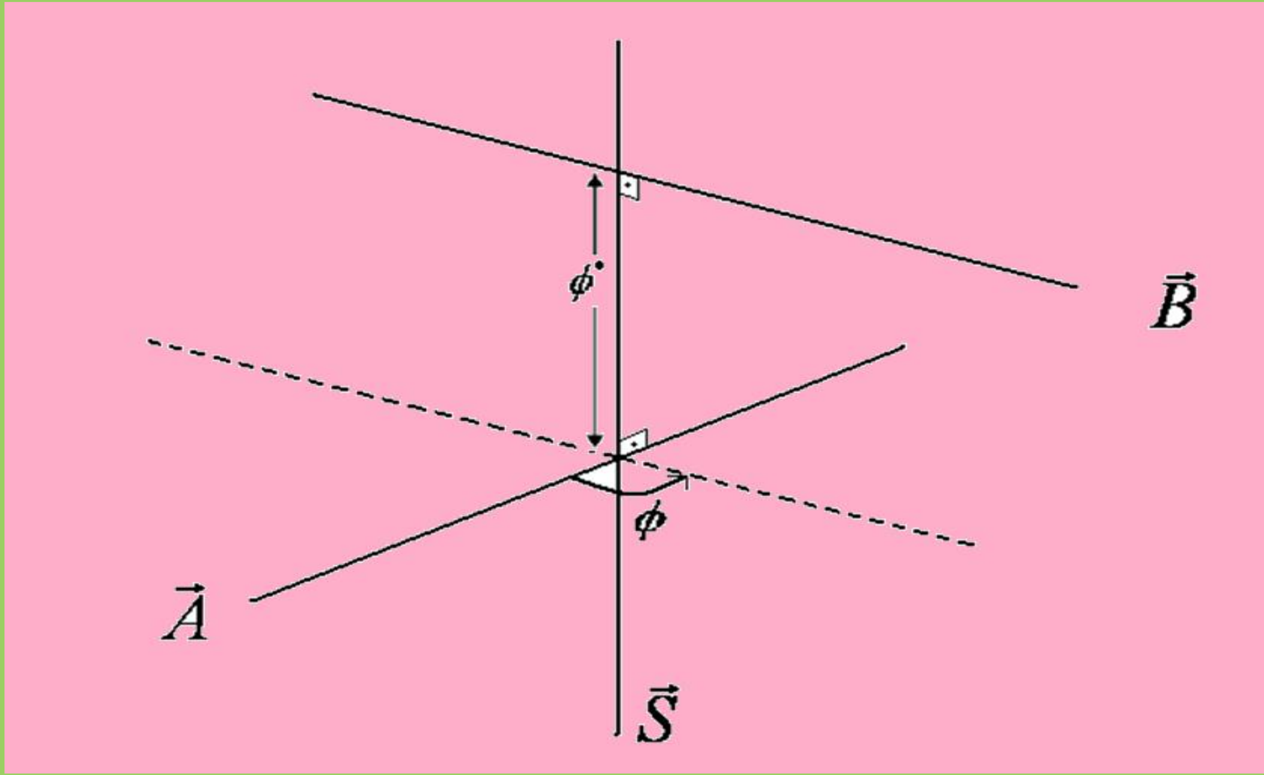
We have form experession (3)

$$\vec{B}\vec{A} = -\cos\Phi - \sin\Phi\vec{S}$$
$$= -(\cos\Phi + \sin\Phi\vec{S})$$

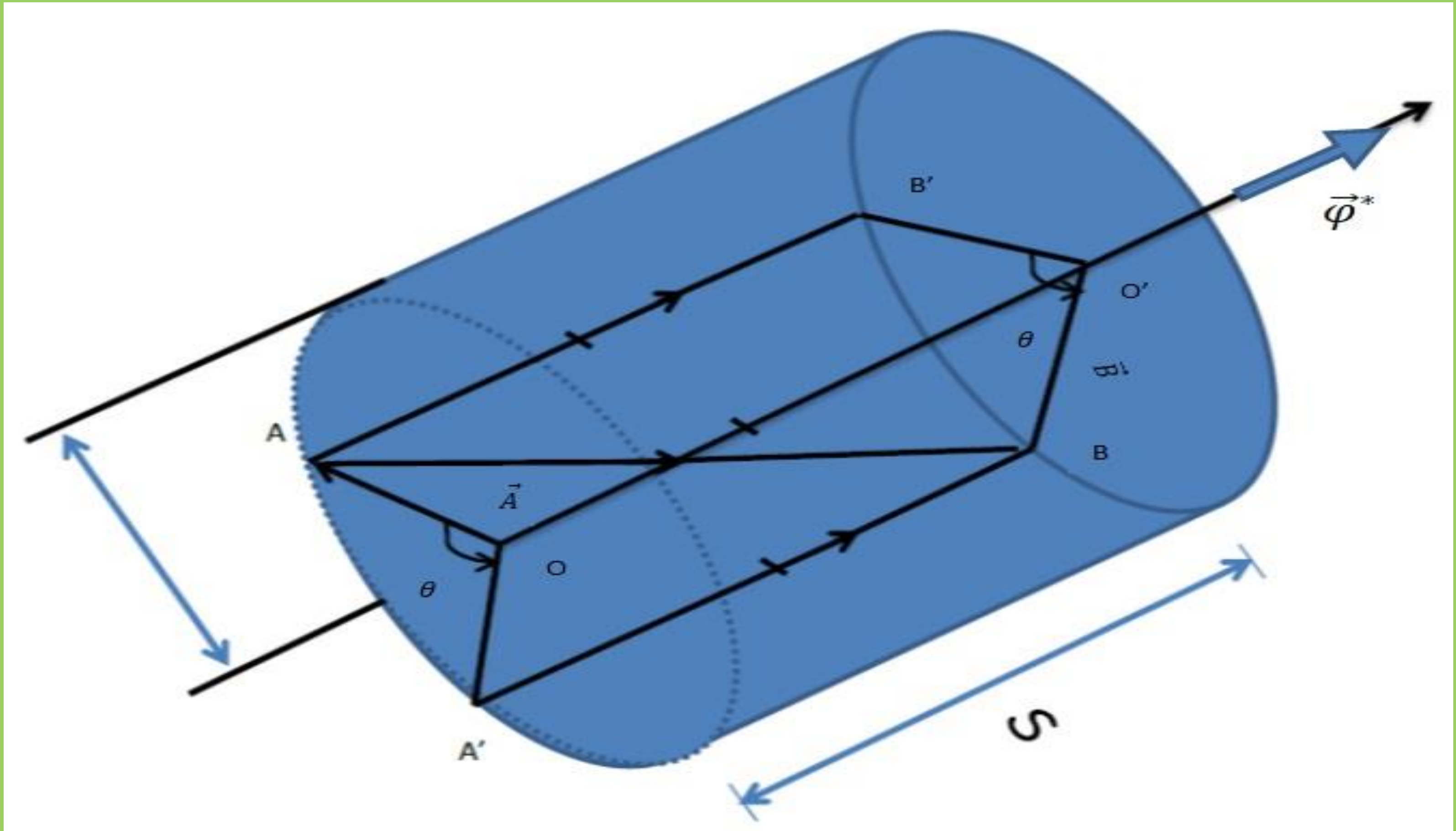
or

$$\vec{B}\vec{A}^{-1} = \cos\Phi + \sin\Phi\vec{S} = Q \quad , \quad \vec{B} = Q\vec{A} \qquad (4)$$

in equation (4),



$\phi^{\bullet}$
$\vec{B}$
ϕ
$\vec{A}$
$\vec{S}$



B′
$\vec{\varphi}^*$
O′
θ
$\vec{B}$
A
B
$\vec{A}$
O
θ
A′
S

RESULT:

In ID_s^3 the left product of a unit vector $\vec{A}$ by an unit dual quaternion Q means that line corresponding to $\vec{A}$ rotating about $\vec{S}$ in the amount of ϕ in positive direction and translation about $\vec{S}$ in the amount of ϕ^* in positive direction.

Special Cases

1) If $\phi \neq 0,\ \phi^{*} = 0$ then

$$Q_1 = \cos\phi + \sin\phi\vec{S}$$

is rotating operator.

2) If $\phi = 0,\ \phi^{*} \neq 0$ then

$$Q_2 = 1 + \varepsilon\phi^{*}\vec{S}$$

is translation operator.

Theorem: A screw operator Q is composed of a rotating operator Q_1 and translation operator Q_2. In other words,

$$Q = Q_1 Q_2 = Q_2 Q_1$$

Proof:

$$\begin{aligned} Q_1 Q_2 &= (\cos\phi + \sin\phi \vec{S})(1 + \varepsilon\phi^* \vec{S}) \\ &= \cos\Phi + \sin\Phi\,\vec{S} \\ &= Q \end{aligned}$$

Frobenius' Theorem

If A is an associative division algebra over the field of reals $\mathbb{R}$ then A is isomporphic to one of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$.

A theorem due to G. Frobenius shows that a 3-dimensional algebra of the type originally sought by Hamilton cannot exist [17, p. 327]. However, if we do not require the structure to be an algebra then it is possible to define a non-trivial commutative multiplication of number triplets. It can be done by using a 3-dimensional subspace of the quaternion algebra and identifying number triplets with quaternions in this subspace. The 3-space of triplets can be considered as a collection of complex planes containing a common line, somewhat like a Rolodex file whose cards are the complex planes and whose axle is the given line.

Frank R. Pfaff became intrigued by this problem of multiplying triplets 60 years ago. In June 1939, as a newly graduated engineer from the University of Notre Dame with an interest in mathematics, He entered the graduate mathematics program at the University, quite unaware of the distinction between undergraduate mathematics courses for engineers and the "real thing".

Professor Karl Menger, who was chairman of Notre Dame's Mathematics Department, felt that Frank R. Pfaff could rise to the challenge, and he recommended that prior to my formal enrollment as a graduate student Frank R. Pfaff should read E. T. Bell's Men of Mathematics. He later asked him to drive him up to the Upper Peninsula of Michigan, where he stayed for two months by himself with a full trunkload of well used texts in a single-room, sparsely furnished cabin and enjoyed a prolonged period of solitary study. During the rather lengthy drive he discussed several math topics with him.

In one such discussion he referred to Hamilton's work and mentioned that no one had ever devised a meaningful commutative multiplication for number triplets. This observation must have stayed with him subliminally for 55 years, because shortly after my wife's death the problem of finding such a multiplication arose and with such persistence that Frank R. Pfaff struggled with it continuously until this present paper emerged.

Although Frank R. Pfaff passed his courses in the fall semester of 1939, his confidence in his ability to complete graduate work for a degree was shaken, and Frank R. Pfaff left the University of Notre Dame in January 1940 to take a job as an engineer with Exxon Corporation. During his early years there Frank R. Pfaff became interested in using the newly emerging computer technology to apply linear programming and statistics to the petroleum industry. This work helped to restore his confidence in his ability to learn, apply, and create mathematics. Thus, this present endeavor arrives after a hiatus of almost three score years.

Frank R. Pfaff show how to define a commutative multiplication of number triplets using a 3-dimensional subspace of the space of quaternions. It turns out that there is a unit element, that every non-zero triplet has a unique multiplicative inverse, and, therefore, that division is possible. Moreover, there are no divisors of zero. However, the multiplication does not obey the distributive law so our resulting structure is not a linear algebra (much less a division algebra).

He recall that in the algebra of quaternions an orthonormal basis in the underlying vector space is denoted by 1, i, j, and k. The quaternion multiplication, X, of these elements is given by Table 1:

X	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

Table 1. Quaternion Multiplication

Using this basis, an arbitrary quaternion can be written as

$$Q = x1 + yi + zj + wk, \qquad\qquad x, y, z, w \in \mathbb{R},$$

or, setting

$$1 = (1,0,0,0), \quad i = (0,1,0,0), \quad j = (0,0,0,1), \quad k = (0,0,0,1),$$

as

$$Q = (x, y, z, w).$$

The (non-commutative) multiplication of quaternions is then defined using Table 1 and the distributive and associative laws.

Those quaternions whose fourth component is zero will consider only , i.e., quaternions of the form

$$X = x1 + yi + zj + 0k = (x, y, z, 0).$$

The set of all such *truncated quatemions* forms a 3-dimensional vector subspace of the space of quaternions, and it is spanned by the (mutually orthogonal) quaternion elements 1, i, and j. Each quadruplet (x, y, z, 0) by the corresponding triplet (x, y, z) replace so that 1 = (1, 0, 0), i = (0, 1, 0), and j = (0, 0, 1), and a general truncated quaternion is written as

$$X = x1 + yi + zj.$$

Of course, this subspace is not closed with respect to quaternion multiplication. However, they defined a (commutative) multiplication for which the subspace of triplets is closed. They next wish to consider the set of all planes, in this subspace, containing the x-axis; somewhat like the set of all cards in a Rolodex file (except that the entire plane is used) with the x-axis as the axle. They refer to any such plane as a leaf. Each leaf is determined by its line of intersection with the (y, z)-plane, or, equivalently, by any non-zero vector U in the (y, z)-plane along this line.

MULTIPLICATION IN A LEAF

Let X=(a,b,c), where b and c are not both zero. Then

$$X=(a,0,0)+(0,b,c)=(a,0,0)+U,$$

where U=(0,b,c) is a non-zero vector in the (y,z)-plane. Thus, U and therefore, X, determines a unique leaf.

A vector Y=(d,e,f)=(d,0,0)+(0,e,f)=(d,0,0)+V, where V=(0,e,f), is in the same leaf as X if and only if there is a real number, say m, such that V=mU, i.e., such that Y=(d,e,f)=(d,mb,mc). Rewriting X and Y as truncated quaternions,

$$X = a1 + bi + cj, \quad Y = d1 + mbi + mcj,$$

We have

$$X \times Y = (a1 + bi + cj) \times (d1 + mbi + mcj)$$
$$= (ad - mb^2 - mc^2)1 + (am + d)bi + (am + d)cj$$
$$+ bmci \times j + cmbj \times i.$$

Then

$$X \times Y = (ad - mb^2 - mc^2)1 + (am + d)bi + (am + d)cj,$$

since the terms in $i \times j$ and $j \times i$ cancel; the product $Z = X \times Y$ is a truncated quaternion.

Multiplication of triplets in a leaf has the following properties:

1) The vector $Z = X \times Y$ is in the leaf of X and Y, i.e., each leaf is closed with respect to this multiplication.
2) The multiplication is commutative.
3) The multiplication is associative and distributive for vectors in the same leaf, since this multiplication is a special case of quaternion multiplication.
4) If $Y = (a, -b, -c)$ then $X \times Y = (a^2 + b^2 + c^2)1 = \|X\|^2 1$, where $\|X\|^2 = \sqrt{X.X}$ is the euclidean norm of X. The quaternion $X^* \equiv a1 - bi - cj$ is just the quaternion conjugate of X; it lies in the leaf of X.
5) If X is a non-zero vector then $X^{-1} = X^* / \|X\|^2$ satisfies $X^{-1} \times X = X \times X^{-1} = 1$, so X^{-1} is the multiplicative inverse of X; it also lies in the leaf of X.

6) There are no divisors of zero since each non-zero triplet has a unique inverse.
7) Finally, for any triplets X and Z in the same leaf the equation $X \times Y = Z$ has a unique triplet solution Y in the leaf of X and Z, namely, $Y = X^{-1} \times Z$.

We see that as long as we restrict ourselves to a leaf, quaternion multiplication of truncated quaternions yields a commutative and associative division algebra on the leaf. Since a leaf is 2-dimensional this gives a complex structure on each leaf in a natural way [18].

1. Introduction

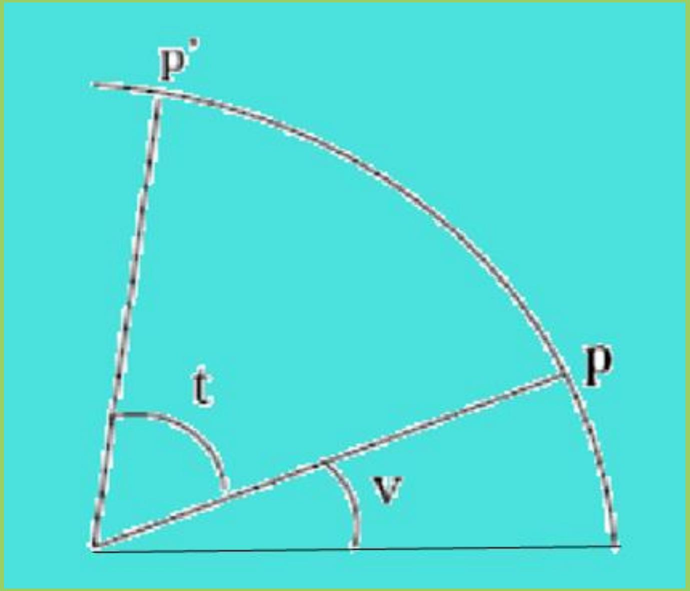
2. Hamilton Operators and their Properties

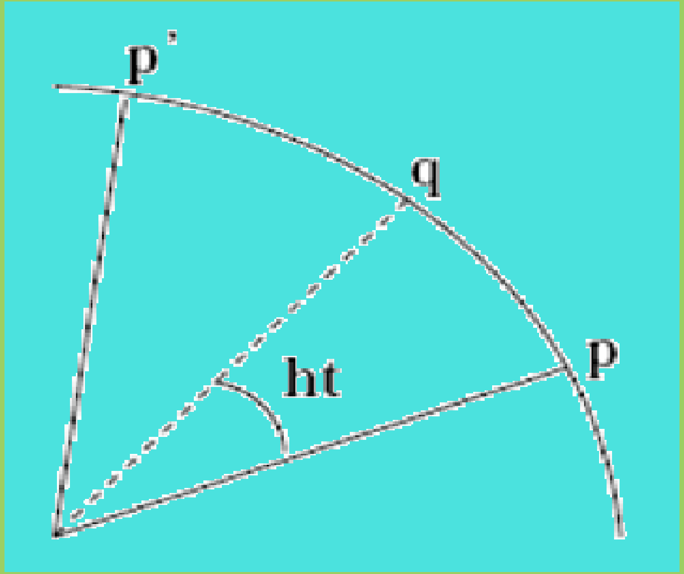
3. Applications of Quaternions

APPLICATIONS OF QUATERNIONS

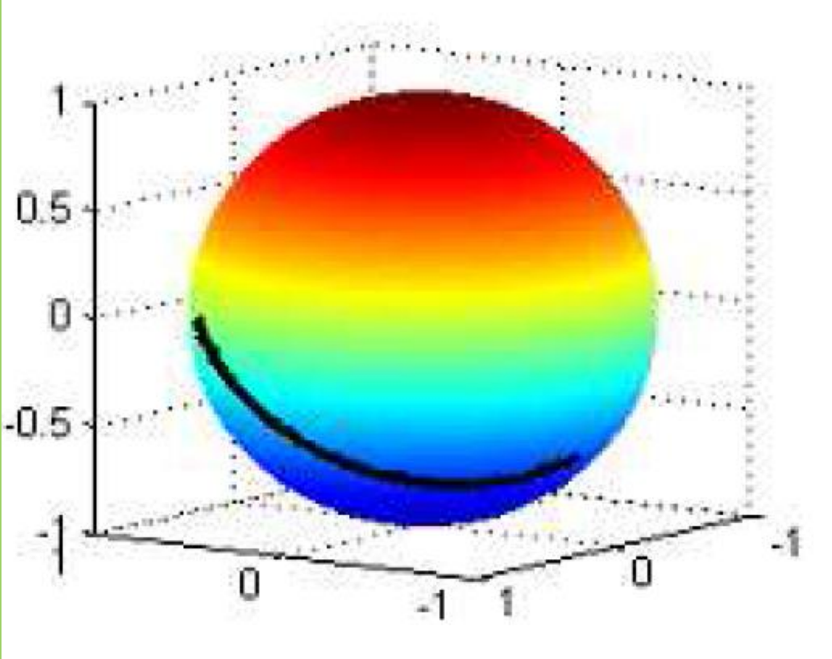
LINEAR INTERPOLATION IN MINKOWSKI SPACE

In computer graphics, spherical linear interpolation (slerp) is shorth and for spherical linear interpolation, in the context of quaternion interpolation for the purpose of animating 3D rotation [Shoemake]. Linear interpolation have been done on sphere Euclidean using quaternions. The linear interpolation on Loretzian sphere Minkowski space have been done using split quaternions. That also yields the shortest possible interpolation path between the two split quaternion on the unit Lorentzian sphere [5].





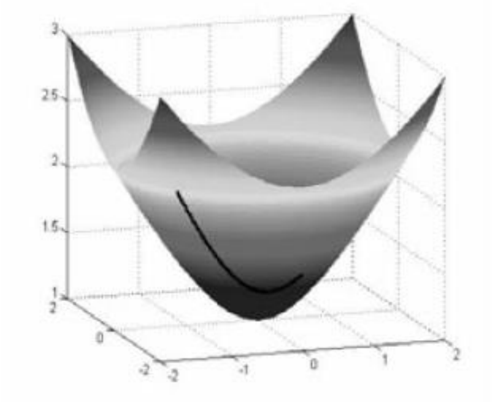
Slerp in the plane: a) The interpolation goes from p to p ' across the angle t; b) A step in the interpolation, where ($h \in [0, 1]$), q moves from p to p '.



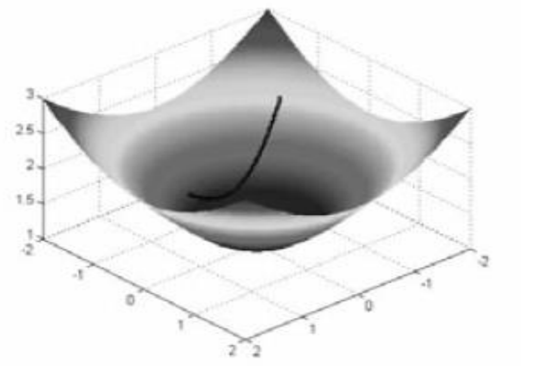
The shapes of interpolation are simulated with MATLAB R2010a: a) Quaternion interpolation between the two key frames in Euclidean space, there are 50 interpolated frames; b) Velocity graph quaternion interpolation.

SPLINE SPLIT QUATERNION INTERPOLATION

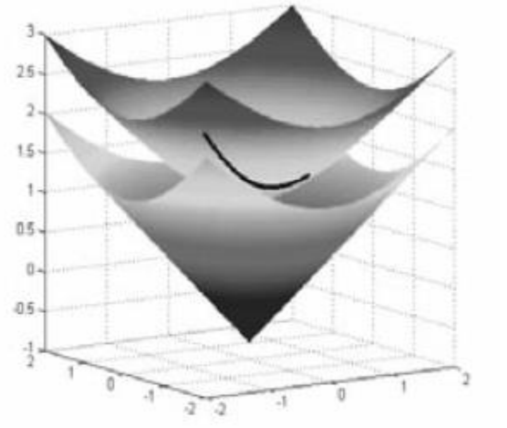
Spherical spline quaternion interpolation has been done on sphere in Euclidean space using quaternions. The spline split quaternion interpolation on hyperbolic sphere in Minkowski space has been done using split quaternions and metric Lorentz. This interpolation curve is called spherical spline split quaternion interpolation in Minkowski space (MSquad) [6].



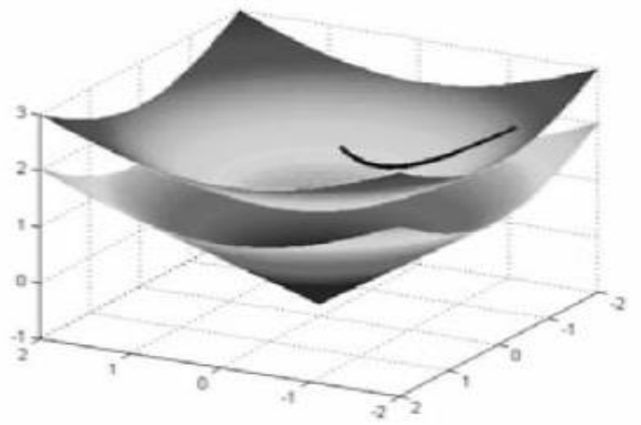
(a)



(b)

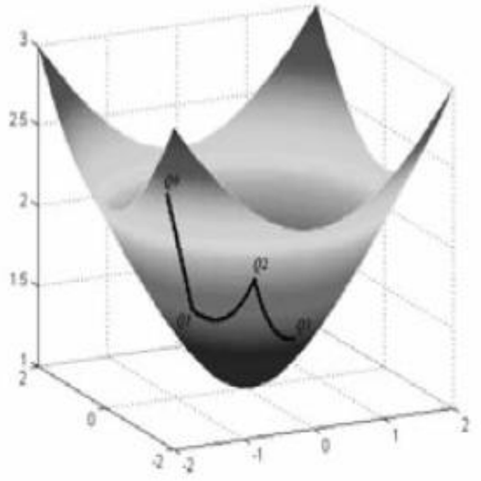


(c)

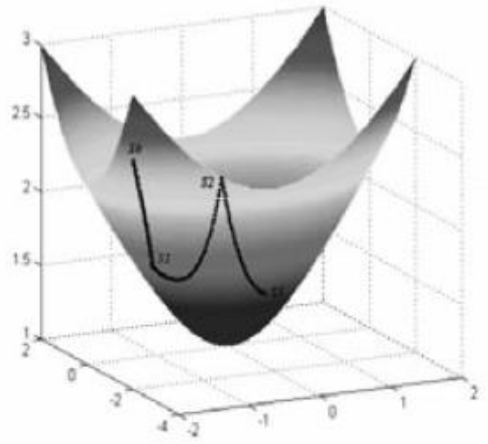


(d)

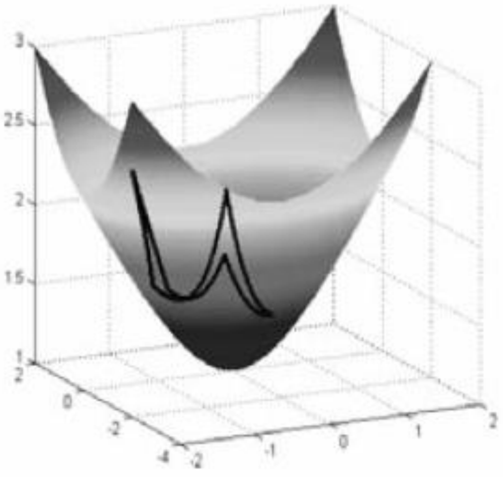
The shapes of interpolation are simulated with MATLAB Programming Language.
a) The interpolation curve between two split quaternion on hyperbolic sphere in Minkowski space, there are 50 interpolated frames; b) Inside scope; c) Outside scope; d) Inside scope.



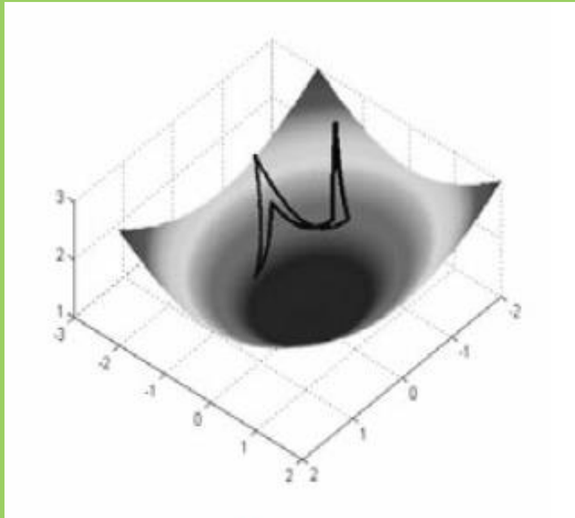
(a)



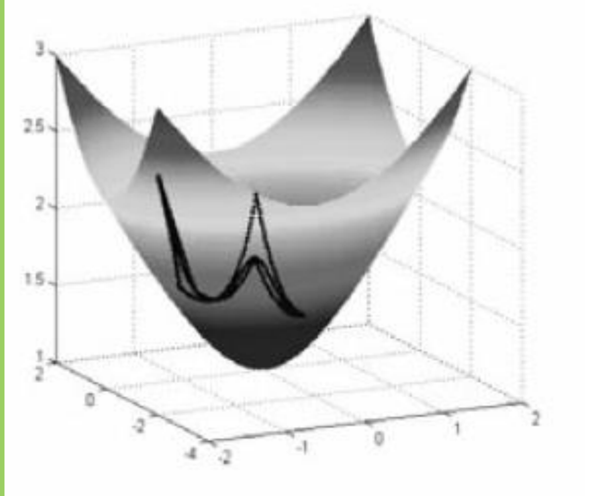
(b)



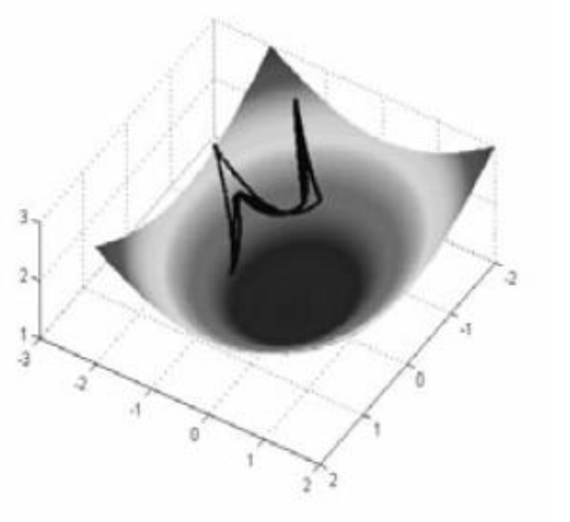
(c)



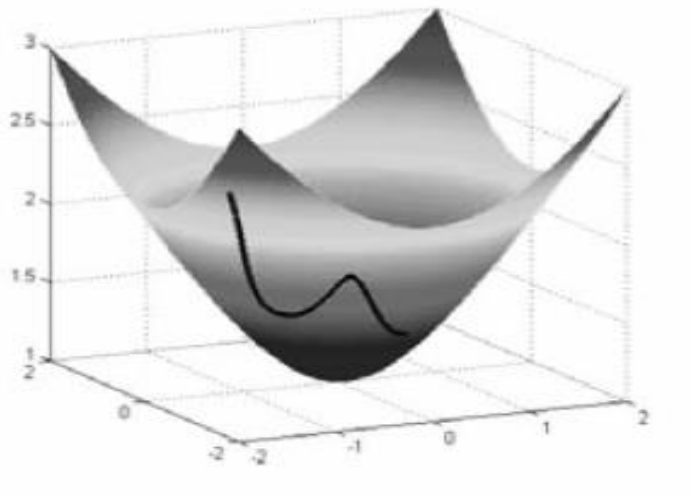
(d)



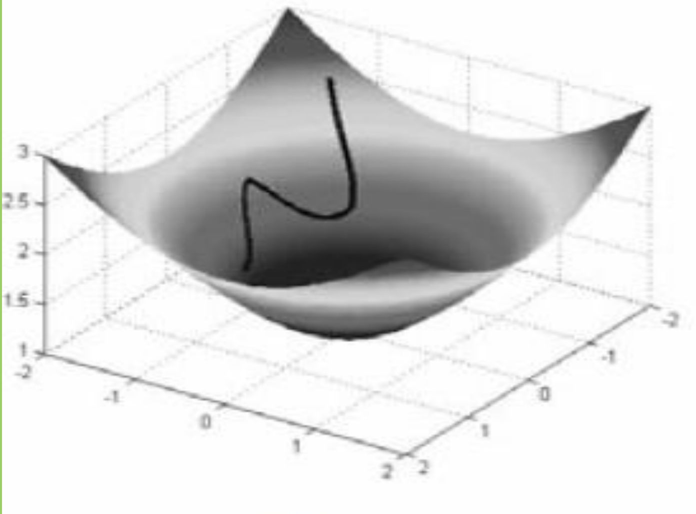
(e)



(f)



(g)

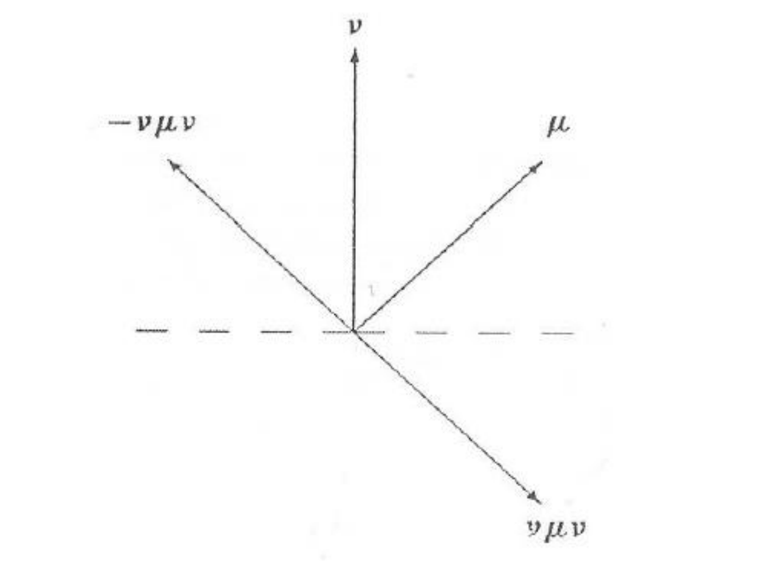


(h)

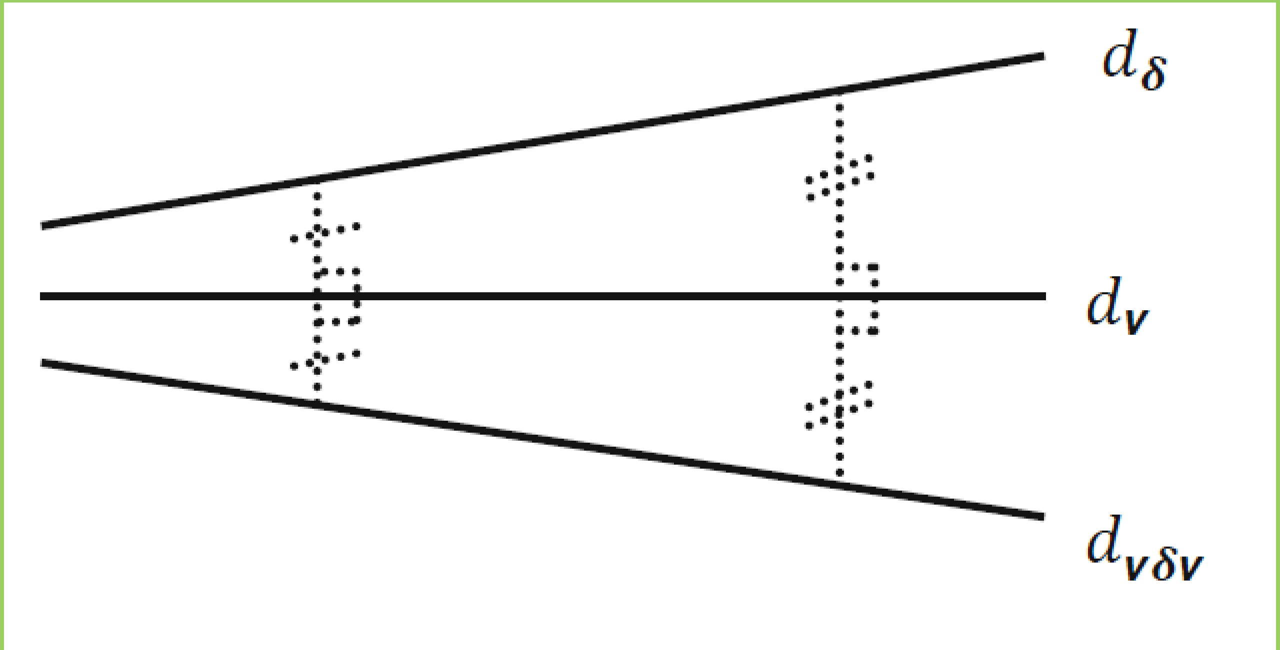
Split quaternion interpolation between the four key frames on hyperbolic sphere; (b) Inner quadrangle interpolation between the four key frames on hyperbolic sphere; (c) Combination of split quaternion and inner quadrangle on the hyperbolic sphere; (d) Inside scope Combination of split quaternion and inner quadrangle on the hyperbolic sphere; (e) Smoothing split quaternion with using inner quadrangle; (f) Inside scope Smoothing split quaternion with using inner quadrangle, (g) interpolation curve for MSquad; (h) Inside scope interpolation curve for Msquad.

Dual Quaternion Involutions and Anti-Involutions

An involution or anti-involution is a self-inverse linear mapping. Involutions and anti-involutions of dual quaternions are shown. Geometric interpretations of real quaternion and dual quaternion involutions and anti-involutions are given. Also, their geometric interpretations are given as reflections [14].



Geometry of the involution $f_v(q) = -v\bar{q}v$ *and* anti-involution
fv(q) = −vqv, where q = a+μb. The dotted line represents a plane perpendicular to *v seen edge-on. (The* scalar part of *q is invariant and not represented in the figure.)*



Geometry of the transformation $v\delta v$ in R^3. d_δ , d_v and $d_v\delta_v$ represent the lines corresponding to unit pure dual quaternions δ, v, and $v\delta_v$ in R^3, respectively.

Quaternion Frenet Frames

The Frenet and parallel-transport frames are reformulate in terms of quaternions only. A quaternion frame is a unit-length four-vector $\mathrm{q} = (q_0, q_1, q_2, q_3) = (q_0, \vec{q})$ that corresponds to exactly one 3D coordinate frame and is characterized by the following properties:

- Unit Norm. The components of a unit quaternion obey the constraint,

$$(q_0)^2 + (q_1)^2 + (q_2)^2 + (q_3)^2 = 1$$

and therefore lie on S3, the three-sphere.

- Multiplication rule. Two quaternions q and p obey the following multiplication rule, which is isomorphic to multiplication in the group SU(2), which is the double covering of the ordinary 3D rotation group SO(3):

$$q.p = \begin{Bmatrix} [q.p]_0 \\ [q.p]_1 \\ [q.p]_2 \\ [q.p]_3 \end{Bmatrix} = \begin{Bmatrix} q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3 \\ q_0p_1 + p_0q_1 + q_2p_3 - q_3p_2 \\ q_0p_2 + p_0q_2 + q_3p_1 - q_1p_3 \\ q_0p_3 + p_0q_3 + q_1p_2 - q_2p_1 \end{Bmatrix}$$

- Inverse. The inverse quaternion is de_ned as

$$\bar{q} = q^{-1} = (q_0, -\vec{q}),$$

so that

$$\bar{q}q = q\bar{q} = (1,\ \vec{0}).$$

- Mapping to 3D rotations. Every possible 3D rotation R (a 3x3 orthogonal matrix) can be constructed from either of two related quaternions, $\mathrm{q} = (q_0, q_1, q_2, q_3)$ or $-\mathrm{q} = (-q_0, -q_1, -q_2, -q_3)$, using the transformation law:

$$[q.\vec{V}.\bar{q}]_i = \sum_{j=1}^{3} R_{ij} V_j$$

where, with $\mathrm{v} = (0,\ \vec{V})$ a pure 3-vector, the quadratic formula of R_{ij} ;

$$R = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0\,q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

[16].

$$R = \begin{array}{c} \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \\ \begin{array}{ccc} \mathrm{T} & \qquad\qquad\qquad \mathrm{N} \qquad\qquad\qquad & \mathrm{B} \end{array} \end{array}$$

A non rotating quaternionic frame is obtained as shown above.

Canal Surfaces with Quaternions

Quaternions are more usable than three Euler angles in the three dimensional Euclidean space. Thus, many laws in different fields can be given by the quaternions. Canal surfaces and tube surfaces can be obtained by the quaternion product and by the matrix representation. Also, the equation of canal surface given by the different frames of its spine curve can be obtained by the same unit quaternion. In addition, these surfaces are obtained by the homothetic motion [15].

AN EXAMPLE

For unit speed curve $\alpha = (\frac{t}{2}\,,\ sin\frac{\sqrt{3}t}{2},\ cos\frac{\sqrt{3}t}{2}),$ the Frenet frame vectors can be given as

$$T(t) = (\frac{1}{2}, \frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}t}{2}, -\frac{\sqrt{3}}{2}\ sin\frac{\sqrt{3}t}{2}),$$

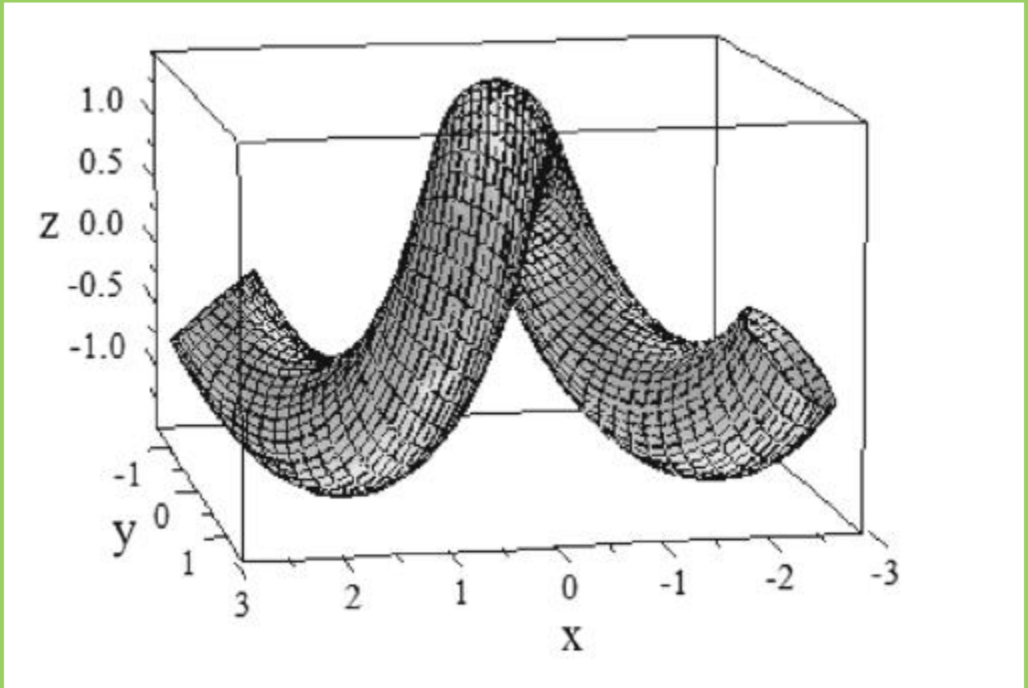
$$N(t) = (0, -\sin\frac{\sqrt{3}t}{2}, -\ cos\frac{\sqrt{3}t}{2}),$$

$$B(t) = (-\frac{\sqrt{3}}{2}, \frac{1}{2}\cos\frac{\sqrt{3}t}{2}, -\frac{1}{2}\ sin\frac{\sqrt{3}t}{2}).$$

Then, for the unit quaternion $q(t,\theta) = cos\theta + sin\theta\ T(t),$ the matrix representation M of ϕ as

$$\begin{bmatrix} \cos^2\theta-\frac{1}{2}\sin^2\theta & \sqrt{3}\sin\theta(\sin\frac{\sqrt{3}t}{2}\cos\theta+\frac{1}{2}\sin\theta\cos\frac{\sqrt{3}t}{2}) & \sqrt{3}\sin\theta(\cos\theta\cos\frac{\sqrt{3}t}{2}-\frac{1}{2}\sin\frac{\sqrt{3}t}{2}\sin\theta) \\ \sqrt{3}\sin\theta(\frac{1}{2}\sin\theta\cos\frac{\sqrt{3}t}{2}-\sin\frac{\sqrt{3}t}{2}\cos\theta) & \cos^2\theta+\sin^2\theta(\frac{3}{4}\cos^2\frac{\sqrt{3}t}{2}-\frac{1}{4}-\frac{3}{4}\sin^2\frac{\sqrt{3}t}{2}) & -\sin\theta(\frac{3}{2}\sin\frac{\sqrt{3}t}{2}\sin\theta\cos\frac{\sqrt{3}t}{2}+\cos\theta) \\ -\sqrt{3}\sin\theta(\frac{1}{2}\sin\frac{\sqrt{3}t}{2}\sin\theta-\cos\theta\cos\frac{\sqrt{3}t}{2}) & \sin\theta(\cos\theta-\frac{3}{2}\sin\frac{\sqrt{3}t}{2}\sin\theta\cos\frac{\sqrt{3}t}{2}) & \cos^2\theta+\sin^2\theta(\frac{3}{4}\sin^2\frac{\sqrt{3}t}{2}-\frac{3}{4}\cos\frac{\sqrt{3}t}{2}-\frac{1}{4}) \end{bmatrix}$$

$$\begin{aligned} X(t,\theta) &= \alpha(t)+rMN(t) \\ &= \alpha(t)+rqxN \\ &= \begin{pmatrix} \frac{t}{2}-r\frac{\sqrt{3}}{2}\sin 2\theta, \sin\frac{\sqrt{3}t}{2}-r\cos 2\theta\sin\frac{\sqrt{3}t}{2}+r\frac{1}{2}\sin 2\theta\cos\frac{\sqrt{3}t}{2}, \\ \cos\frac{\sqrt{3}t}{2}-r\cos 2\theta\cos\frac{\sqrt{3}t}{2}-r\frac{1}{2}\sin 2\theta\sin\frac{\sqrt{3}t}{2}, \\ \cos\frac{\sqrt{3}t}{2}-r\cos 2\theta\cos\frac{\sqrt{3}t}{2}-r\frac{1}{2}\sin 2\theta\sin\frac{\sqrt{3}t}{2} \end{pmatrix} \end{aligned}$$



Tube Surface $X(t,\theta)$

Curvature of Almost Split Quaternion Kaehler Manifolds

Quaternion Kaehler Manifolds are frequent studied a subject. It is important study some characterizations of Riemann curvature and Ricci curvature on quaternion Kaehler manifolds. Split quaternions are a new developing topic. Inoguchi, J. studied on this topic. These characterizations of Riemann curvature and Ricci curvature on split quaternion Kaehler manifolds [10].

From Golden Spirals to Constant Slope Surfaces

All constant slope surfaces in the Euclidean 3-space are found. Namely, those surfaces for which the position vector of a point of the surface makes constant angle with the normal at the surface in that point. These surfaces could be thought as then bi-dimensional analogue of the generalized helices [7].

AN EXAMPLE

Let us point more attention to this picture (but not necessary with $\theta = \frac{\pi}{5}$), when $f(\mathrm{v}) = (\cos\mathrm{v}, \sin\mathrm{v}, 0)$. Then,

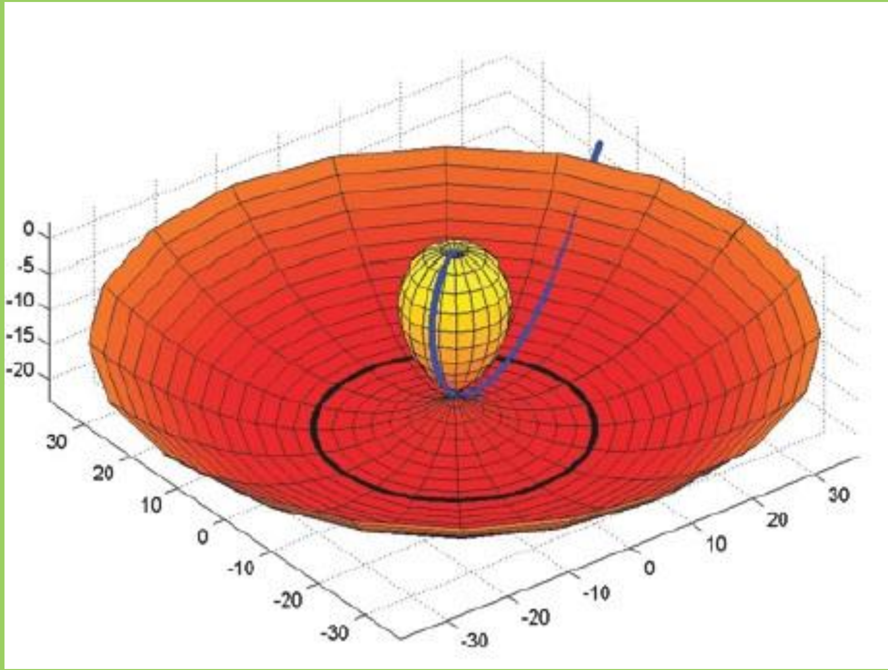
$$f(\mathrm{v})\,\mathrm{x}\,f'(\mathrm{v}) = (0,0,1)$$

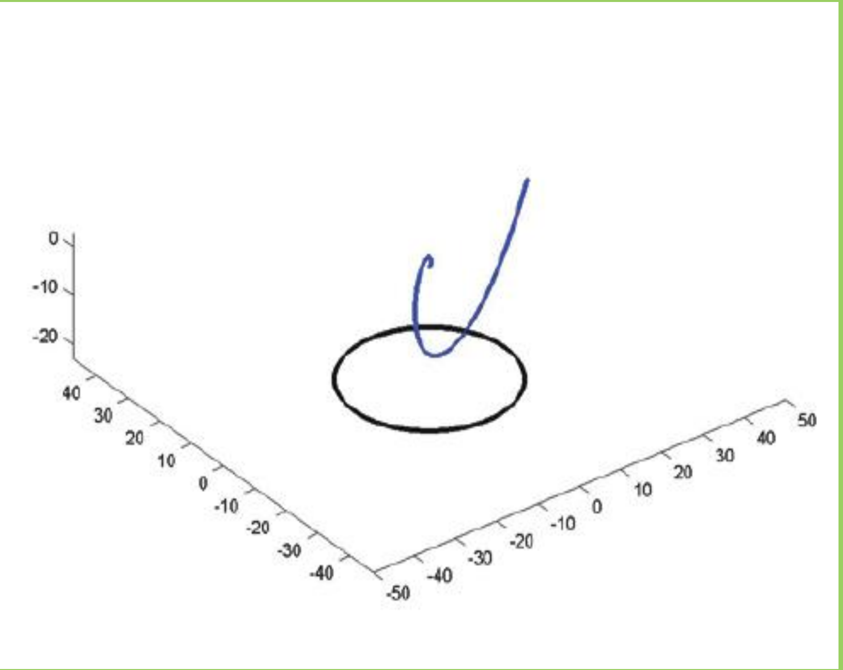
for all v and consequently the slope surface is parametrized by

$$\mathrm{r}(\mathrm{u},\mathrm{v}) = u\sin\theta(\cos(\xi(\mathrm{u}))\cos\mathrm{v}, \cos(\xi(\mathrm{u}))\sin\mathrm{v}, \sin(\xi(\mathrm{u}))),$$

$$\mathrm{Q} = u\sin\theta(\cos\xi - \sin\xi\, f'),$$

$$\mathrm{Q}\mathrm{x}\; f = \mathrm{r}(\mathrm{u},\mathrm{v}).$$





For $\theta = \dfrac{\pi}{5}$, $f(\mathrm{v}) = (\cos\mathrm{v}, \sin\mathrm{v}, 0)$

On the quaternionic Mannheim curves of $Aw(k)$-type in Euclidean space E^3

The curvature conditions of $Aw(k)$-type $(1\leq k \leq 3)$ quaternionic curves in Euclidean space E^3 and quaternionic Mannheim curves $\alpha : I \rightarrow Q$ with $k \neq 0$ and $r \neq 0$ are shown . Besides, quaternionic Mannheim curves are $Aw(2)$ -type and Aw (3)-type quaternionic curves in E^3. But, there is no such a Mannheim curve of $Aw(1)$ –type [13].

Homothetic motions at E^4

A Hamilton motion has been defined in four-dimensional Euclidean space E^4, and it is shown that this is a homothetic motion. Furthermore, it has been found that the Hamilton motion defined by a regular curve of order r has only one acceleration centre of order $(r-1)$ at every t-instant [12].

HOMOTHETIC MOTIONS AT E^8 WITH CAYLEY NUMBERS

A matrix which is similar to Hamilton operators has been developed for Cayley numbers in eight dimensional Euclidean space E^8 and a new motion has been defined by this matrix. It is shown that this is a homothetic motion. Furthermore, it has been found that the motion defined by
regular curve of order r has only one acceleration centre of order (r - 1) at every instant [11].

Circular Surfaces with Split Quaternionic Representations in Minkowski 3-space

Circular surfaces are smooth one-parameter families of circles. Three main purposes about circular surfaces and roller coaster surfaces are defined as circular surfaces whose generating circles are lines of curvature. The first one is to reconstruct equations of spacelike circular surfaces and spacelike roller coaster surfaces by using unit split quaternions and homothetic motions.

The second one is to parametrize timelike circular surfaces and give some geometric properties such as striction curves, singularities, Gaussian and mean curvatures. Furthermore, the conditions for timelike roller coaster surfaces to be flat or minimal surfaces are obtained. The last one is to express split quaternionic and matrix representations of timelike circular surfaces and timelike roller coaster surfaces.

AN EXAMPLE

Given a curve

$$\xi(s) = \left(\frac{5s}{12}, \frac{4}{9}\cos\left(\frac{3s}{4}\right), \frac{4}{9}\sin\left(\frac{3s}{4}\right)\right),$$

it is easy to show that

$$a_1(s) = e(s) = \left(\frac{5}{3}, -\frac{4}{3}\sin\left(\frac{3s}{4}\right), \frac{4}{3}\cos\left(\frac{3s}{4}\right)\right) \triangleleft$$

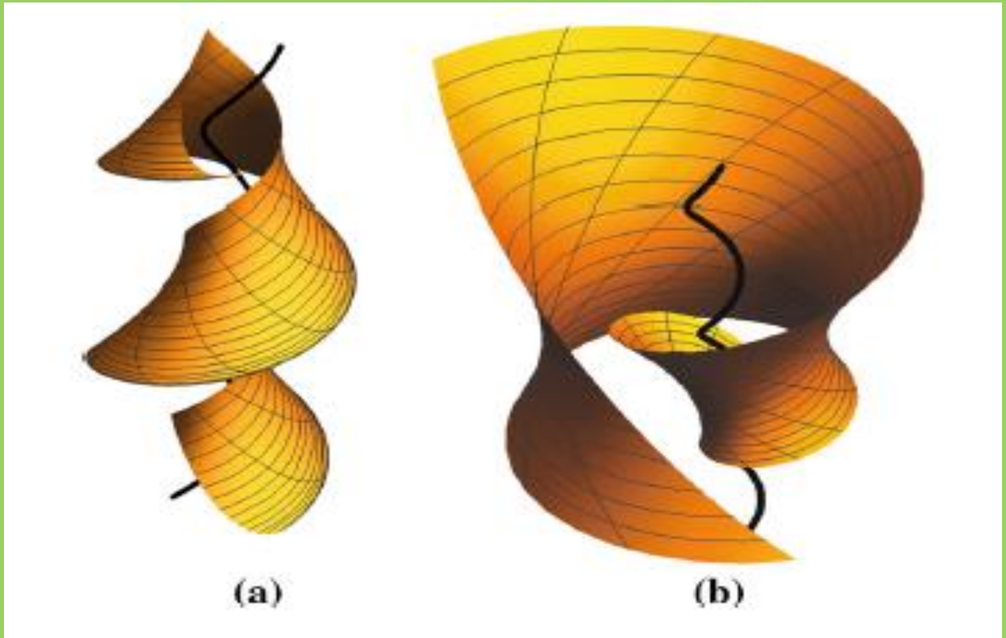
$$a_2(s) = n(s) = \left(0, -\cos\left(\frac{3s}{4}\right), -\sin\left(\frac{3s}{4}\right)\right)$$

$$a_3(s) = b(s) = \left(-\frac{4}{3}, \frac{5}{3}\sin\left(\frac{3s}{4}\right), -\frac{5}{3}\cos\left(\frac{3s}{4}\right)\right)$$

where *s* is the arc-length parameter of $a_1(s)=e$. Using the timelike unit split quaternion $q_e=\cos\theta+e\sin\theta$, we have

$$C_{(\xi,n,b,r)}(s,\theta)=\xi(s)+r(s)q_e(s,\theta)*n(s).$$

For r = 1 and r = $s/2$, the spacelike circular surfaces are illustrated.



The spacelike circular surfaces with r = 1 and r = $s/2$. **a.** The circular surface $C_{(\xi,e,n,1)}$. **b.** The circular surface $C_{(\xi,e,n,s/2)}$.

A New Representation of Canal Surfaces with Split Quaternions in Minkowski 3-Space

Canal surfaces determined by spherical indicatrices of any spatial curve in Minkowski 3-space by means of timelike split quaternions. Moreover, using orthogonal matrices corresponding to these quaternions, the canal surfaces are obtained as homotetic motions. Then, we investigate a relationship between the canal surfaces and unit split quaternions [8].

AN EXAMPLE

Given a unit-speed spacelike curve

$$\alpha(s)=\left(\frac{4}{15}\sin 5s,\ -\frac{1}{24}\cos 8s+\frac{2}{3}\cos 2s,\ \frac{1}{24}\sin 8s+\frac{2}{3}\sin 2s\right)$$

With spacelike binormal vector, the alternative moving frame vectors are given by

$$N(s)=\left(-\frac{5}{3},\ -\frac{4}{3}\sin 3s,-\frac{4}{3}\cos 3s\right),$$

$$C(s)=(0,-\cos 3s,\sin 3s),$$

$$W(s)=\left(\frac{4}{3},\frac{5}{3}\sin 3s,\frac{5}{3}\cos 3s\right).$$

Then, the tangent, principal normal and binormal indicatrices of the curve α are obtained as follows:

$$T(\varphi_T(s)) = (\frac{4}{3}\cos 5s,\ \frac{1}{3}\sin 8s - \frac{4}{3}\sin 2s,\ \frac{1}{3}\cos 8s + \frac{4}{3}\cos 2s),$$

$$N(\varphi_N(s)) = (-\frac{5}{3}, -\frac{4}{3}\sin 3s, -\frac{4}{3}\cos 3s),$$

$$B(\varphi_B(s)) = (-\frac{4}{3}\sin 5s, \frac{1}{3}(-4\cos 2s + \cos 8s),$$

$$-\frac{8\cos^3 s}{3}(6\sin s - 3\sin 3s + \sin 5s)).$$

For the unit timelike split quaternion

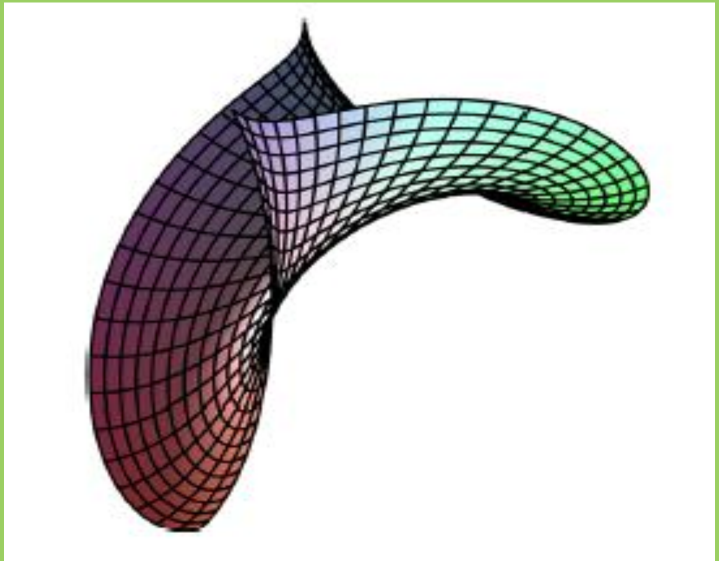
$$q_N(s,\theta) = \cosh\theta + \sinh\theta N$$

with timelike vector part, the parametric equation of the canal surface $P_T(s,\theta)$ constructed by the tangent indicatrix T of the curve α with the admission $r(s) = \sin s$ is found as follows;

$$P_T(s,\theta) = (\frac{4}{3}\cos 5s + \frac{5}{6}\sin 2s + \frac{4}{3}\sin^2 s\sin\theta,$$

$$+\frac{1}{3}\sin 8s - \frac{4}{3}\sin 2s + \frac{2}{3}\sin 2s\sin 3s - \sin^2 s\cos 3s\cos\theta + \frac{5}{3}\sin^2 s\sin 3s\sin\theta,$$

$$+\frac{1}{3}\cos 8s + \frac{4}{3}\cos 2s + \frac{2}{3}\sin 2s\cos 3s + \sin^2 s\sin 3s\cos\theta + \frac{5}{3}\sin^2 s\cos 3s\sin\theta).$$



The canal surface $P_T(s,\theta)$ constructed by the tangent indicatrix T of α

Then, taking $r = 1$ and $r = \theta$, respectively, the equations of the tubular and generalized tubular surfaces $\Lambda T\,(s, \theta)$ and $\Omega T\,(s, \theta)$ generated by the tangent indicatrix T are given by

$$\wedge_T(s,\theta) = (\frac{4}{3}\cos 5s + \frac{4}{3}\sin\theta,$$

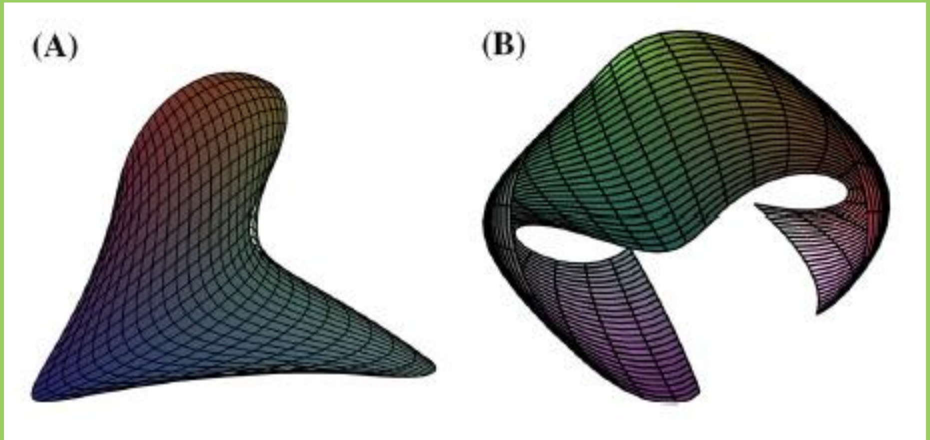
$$\frac{1}{3}\sin 8s - \frac{4}{3}\sin 2s + \cos 3s\cos\theta + \frac{5}{3}\sin 3s\sin\theta,$$

$$\frac{1}{3}\cos 8s + \frac{4}{3}\cos 2s + \sin 3s\cos\theta + \frac{5}{3}\cos 3s\sin\theta),$$

$$\Omega_T(s,\theta) = (\frac{4}{3}\cos 5s + \frac{4}{3}\theta\sin\theta,$$

$$\frac{1}{3}\sin 8s - \frac{4}{3}\sin 2s + \theta\cos 3s\cos\theta + \frac{5}{3}\theta\sin 3s\sin\theta,$$

$$\frac{1}{3}\cos 8s + \frac{4}{3}\cos 2s + \theta\sin 3s\cos\theta + \frac{5}{3}\theta\cos 3s\sin\theta).$$



(A). The tubular surface $\Lambda_T(s,\theta)$ constructed by T.
(B). The generalized tubular surface $\Omega_T(s,\theta)$ constructed by T

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Thank you for your attention